Reachability Analysis of the Equivalence of Two Terms in Free Orthomodular Lattices

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Contents

Word Problem

- The problem of deciding whether or not two given terms (words) of algebras are equivalent is called the **word problem** for the algebras.
- The word problem for various kinds of lattices is one of the central topics in lattice theory.
  - The word problem for **free distributive lattices** is solvable [Takeuchi, 1969].
  - The word problem for **free modular lattices** with $n \leq 3$ generators is solvable and that for free modular lattices with $n \geq 4$ generators is unsolvable [Herrmann, 1983].
  - The word problem for **free ortholattices** is solvable [Bruns, 1976].
  - The word problem for **free modular ortholattices** with $n \leq 2$ generators is solvable and that for general free modular ortholattices remains an open problem [Roddy, 1989].
  - The word problem for **free orthomodular lattices** with $n \leq 2$ generators is solvable and that for general free modular ortholattices remains an open problem [Bruns and Harding, 2000].

Background

- In this presentation, we focus on **free orthomodular lattices** because is significant as the algebraic structure of Quantum Logic.
  - Since 1936, the algebraic structure of quantum mechanics called orthomodular lattices have attracted many logicians' attention [Birkhoff and von Neumann, 1936].
  - This is because the set of all closed subspaces (experimental propositions [Birkhoff and von Neumann, 1936]) of a Hilbert space is an orthomodular lattice [Rédei, 1998, Proposition 4.5].
- However, the word problem for free orthomodular lattices still remains an open problem [Bruns and Harding, 2000].
- Instead of tackling this open problem, we try to implement a tool that supports to check the equivalence of two terms in free orthomodular lattices using Maude.

Previous Studies

- There are two existing programs [Megill and Pavičić, 2001, Hyčko, 2005] for checking the equivalence of two terms in free orthomodular lattices.
- However, these programs cannot deal with terms that consist of three or more free variables.
  - This is because there are infinite normal forms in three or more free generators, even though there are only 96 normal forms in the case of two free generators.
  - This limitation to some two free generators is fatal when proving theorems expressed by three or more generators in orthomodular lattices.
- We overcome this limitation by incorporating the idea of reachability analysis into a neither confluent nor terminating term rewriting system for free orthomodular lattices.

Method

- The word problem is transformed into a **reachability problem** in the term rewriting system by searching for the reachable state space from an initial state.
- The reachability problem is conducted through a breadth-first search in an incremental way, which does not strictly require the reachable state space to be finite.
  (However, the reachability analysis may not terminate in general.)
- Based on this idea, we implement a support tool in Maude, a rewriting logic-based specification/programming language that can deal with terms that consist of three or more free variables.

Result

- Our support tool consists of a formal specification of free orthomodular lattices and an implementation of
  1. the 96 normal forms of two generators and
  2. a theorem describing when the distributive laws can be applied to check the word problem for free orthomodular lattices.
- To demonstrate the effectiveness of our approach, we verify the validity of some axioms with three free variables in several implication algebras [Hardegree, 1981, Abbott, 1976, Chajda et al., 2001, Georgacarakos, 1980].

Contents

Definition 1

A **lattice** is a triple $(L, \wedge, \vee)$ that consists of a non-empty set L and functions $\wedge : L \times L \to L$ and $\vee : L \times L \to L$ satisfying

1. (Associativity) $p \wedge (q \wedge r) = (p \wedge q) \wedge r$ and $p \vee (q \vee r) = (p \vee q) \vee r$,
2. (Commutativity) $p \wedge q = q \wedge p$ and $p \vee q = q \vee p$,
3. (Idempotency) $p \wedge p = p$ and $p \vee p = p$,
4. (Absorption) $p \wedge (p \vee q) = p$ and $p \vee (p \wedge q) = p$.

Definition 2

A lattice $(L, \wedge, \vee)$ is said to be

- **bounded** if it has the least element (denoted by $\curlywedge$) and the greatest element (denoted by $\curlyvee$) under the partial order $\leq$.
- **distributive** if the distributive law holds:

$$p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$$

  for any $p, q, r \in L$.

Definition 3

An **ortholattice** (also called an orthocomplemented lattice) is a bounded lattice equipped with an orthocomplementation. An orthocomplementation on a bounded lattice $(L, \wedge, \vee)$ is a function $\neg : L \to L$ such that

1. $p \wedge \neg p = \curlywedge$ and $p \vee \neg p = \curlyvee$,
2. $\neg\neg p = p$,
3. $\neg(p \wedge q) = \neg p \vee \neg q$ and $\neg(p \vee q) = \neg p \wedge \neg q$,

for any $p, q \in L$. In particular, distributive ortholattices are called **Boolean lattices**.

Definition 4

An **orthomodular lattice** is an ortholattice $(L, \wedge, \vee, \neg)$ satisfying the orthomodular law:

$$p \vee q = p \vee (\neg p \wedge (p \vee q)),$$

for any $p, q \in L$

Definition 5

Let $\mathcal{L}_1 = (L_1, \wedge_1, \vee_1, \neg_1)$ and $\mathcal{L}_2 = (L_2, \wedge_2, \vee_2, \neg_2)$ be ortholatices. Then,

$$\mathcal{L}_1 \times \mathcal{L}_2 = (L_1 \times L_2, \wedge, \vee, \neg)$$

defined by

$$(p_1, p_2) \wedge (q_1, q_2) = (p_1 \wedge_1 q_1, q_1 \wedge_2 q_2),$$
$$(p_1, p_2) \vee (q_1, q_2) = (p_1 \vee_1 q_1, q_1 \vee_2 q_2),$$
$$\neg(p_1, p_2) = (\neg_1 p_1, \neg_2 p_2)$$

is an ortholattice and is called the **direct product** of $\mathcal{L}_1$ and $\mathcal{L}_2$.

Definition 6

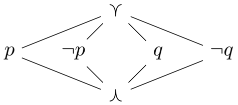
Let $\mathcal{C}$ be a non-empty class of algebras. An algebra $F_X \in \mathcal{C}$ is called a **free algebra** in $\mathcal{C}$ generated by X, if F_X is generated by $X \subseteq F_X$ and every function $V : X \to A \in \mathcal{C}$ can be uniquely extended to a homomorphism $\widehat{V} : F_X \to A$.

- We use two fundamental theorems for orthomodular lattices to implement a support tool.
- The first fundamental theorem states that there are only 96 normal forms in the free orthomodular lattice with two generators.

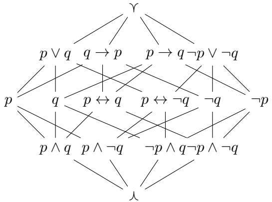
Theorem 1

$MO_2 \times \mathbf{2}^4$ is isomorphic to the free orthomodular lattice with two generators, where MO_2 is a kind of lattice called the Chinese lantern, and $\mathbf{2}^4$ is a free Boolean lattice with two generators.

Proof: see [Beran, 1985, Theorem III.2.8].



Chinese lantern



Free Boolean lattice with two generators p and q

- The second fundamental theorem tells us when the distributive law can be applied in orthomodular lattice.
  (This is a new theorem proposed in this paper.)

Theorem 2

Let $(L, \wedge, \vee, \neg)$ be an orthomodular lattice. Then, the following are equivalent:

1. $p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$;
2. $p \wedge (\neg p \vee q) = p \wedge q$.

Dually, the following are also equivalent:

1. $p \vee (q \wedge r) = (p \vee q) \wedge (p \vee r)$;
2. $p \vee (\neg p \wedge q) = p \vee q$.

Proof: see our paper.

The Role of The Fundamental Theorems

- By Theorem 1, the word problem for free orthomodular lattices with $n \leq 2$ generators is solvable. That is, there are only 96 normal forms (the elements of $MO_2 \times \mathbf{2}^4$ is $6 \times 2^4 = 96$.
  - Note that if p and q are normal forms, then $\neg p$, $\neg q$, $p \wedge q$, and $p \vee q$ are also normal forms.
- By Theorem 2, some terms with three or more variables are simplified using the distributive law under some condition.

96 Normal Forms

$$(p\land q), (p\land\neg q), (\neg p\land q), (\neg p\land\neg q), ((p\land q)\lor(p\land\neg q)), ((p\land q)\lor(\neg p\land q)), ((p\land q)\lor(\neg p\land\neg q)), ((p\land\neg q)\lor(\neg p\land q)), ((\neg p\land\neg q)\lor(p\land\neg q)), ((\neg p\land\neg q)\lor(\neg p\land q)), ((p\land q)\lor(p\land\neg q)\lor(\neg p\land q)), ((p\land q)\lor(p\land\neg q)\lor(\neg p\land\neg q)), ((\neg p\land\neg q)\lor(\neg p\land q)\lor(p\land q)), ((\neg p\land\neg q)\lor(\neg p\land q)\lor(p\land\neg q)), ((p\land q)\lor(p\land\neg q)\lor(\neg p\land q)\lor(\neg p\land\neg q)), (p\land(\neg p\lor q)\land(\neg p\lor\neg q)), (p\land(\neg p\lor q)), (p\land(\neg p\lor\neg q)), ((\neg p\land q)\lor(p\land(\neg p\lor\neg q)\land(\neg p\lor q))), ((\neg p\land\neg q)\lor(p\land(\neg p\lor q)\land(\neg p\lor\neg q))), (p), ((\neg p\lor q)\land(p\lor(\neg p\land q))), ((\neg p\lor q)\land(p\lor(\neg p\land\neg q))), ((\neg p\lor\neg q)\land(p\lor(\neg p\land q))), ((\neg p\lor\neg q)\land(p\lor(\neg p\land\neg q))), ((\neg p\lor\neg q)\land(\neg p\lor q)\land(p\lor(\neg p\land\neg q)\lor(\neg p\land q))), (p\lor(\neg p\land q)), (p\lor(\neg p\land\neg q)), ((\neg p\lor q)\land(p\lor(\neg p\land\neg q)\lor(\neg p\land q))), ((\neg p\lor\neg q)\land(p\lor(\neg p\land q)\lor(\neg p\land\neg q))), (p\lor(\neg p\land q)\lor(\neg p\land\neg q)), (q\land(\neg q\lor p)\land(\neg q\lor\neg p)), (q\land(\neg q\lor p)), ((p\land\neg q)\lor(q\land(\neg q\lor\neg p)\land(\neg q\lor p))), (q\land(\neg q\lor\neg p)), ((\neg p\land\neg q)\lor(q\land(\neg q\lor p)\land(\neg q\lor\neg p))), ((p\lor\neg q)\land(q\lor(\neg q\land p))), (q), ((p\lor\neg q)\land(q\lor(\neg q\land\neg p))), ((\neg p\lor\neg q)\land(q\lor(\neg q\land p))), ((\neg p\lor\neg q)\land(p\lor\neg q)\land(q\lor(\neg p\land\neg q)\lor(p\land\neg q))), ((\neg p\lor\neg q)\land(q\lor(\neg q\land\neg p))), (q\lor(\neg q\land p)), ((p\lor\neg q)\land(q\lor(\neg q\land\neg p)\lor(\neg q\land p))), (q\lor(\neg q\land\neg p)), ((\neg p\lor\neg q)\land(q\lor(\neg q\land p)\lor(\neg q\land\neg p))), (q\lor(\neg q\land p)\lor(\neg q\land\neg p)), (\neg q\land(q\lor\neg p)\land(q\lor p)), ((p\land q)\lor(\neg q\land(q\lor\neg p)\land(q\lor p))), (\neg q\land(q\lor p)), ((\neg p\land q)\lor(\neg q\land(q\lor p)\land(q\lor\neg p))), (\neg q\land(q\lor\neg p)), ((p\lor q)\land(\neg q\lor(q\land p))), ((p\lor q)\land(\neg p\lor q)\land(\neg q\lor(p\land q)\lor(\neg p\land q))), ((\neg p\lor q)\land(\neg q\lor(q\land p))), ((p\lor q)\land(\neg q\lor(q\land\neg p))), (\neg q), ((\neg p\lor q)\land(\neg q\lor(q\land\neg p))), ((p\lor q)\land(\neg q\lor(q\land\neg p)\lor(q\land p))), (\neg q\lor(q\land p)), ((\neg p\lor q)\land(\neg q\lor(q\land\neg p)\lor(q\land p))), (\neg q\lor(q\land\neg p)), (\neg q\lor(q\land\neg p)\lor(q\land p)), (\neg p\land(p\lor\neg q)\land(p\lor q)), ((p\land q)\lor(\neg p\land(p\lor\neg q)\land(p\lor q))), ((p\land\neg q)\lor(\neg p\land(p\lor q)\land(p\lor\neg q))), (\neg p\land(p\lor q)), (\neg p\land(p\lor\neg q)), ((p\lor q)\land(p\lor\neg q)\land(\neg p\lor(p\land q)\lor(p\land\neg q))), ((p\lor q)\land(\neg p\lor(p\land q))), ((p\lor\neg q)\land(\neg p\lor(p\land q))), ((p\lor q)\land(\neg p\lor(p\land\neg q))), ((p\lor\neg q)\land(\neg p\lor(p\land\neg q))), (\neg p), ((p\lor q)\land(\neg p\lor(p\land\neg q)\lor(p\land q))), ((p\lor\neg q)\land(\neg p\lor(p\land q)\lor(p\land\neg q))), (\neg p\lor(p\land q)), (\neg p\lor(p\land\neg q)), (\neg p\lor(p\land\neg q)\lor(p\land q)), ((p\lor q)\land(p\lor\neg q)\land(\neg p\lor q)\land(\neg p\lor\neg q)), ((p\lor q)\land(p\lor\neg q)\land(\neg p\lor q)), ((p\lor q)\land(p\lor\neg q)\land(\neg p\lor\neg q)), ((\neg p\lor\neg q)\land(\neg p\lor q)\land(p\lor q)), ((\neg p\lor\neg q)\land(\neg p\lor q)\land(p\lor\neg q)), ((p\lor q)\land(p\lor\neg q)), ((p\lor q)\land(\neg p\lor q)), ((\neg p\lor q)\land(p\lor\neg q)), ((p\lor q)\land(\neg p\lor\neg q)), ((\neg p\lor\neg q)\land(p\lor\neg q)), ((\neg p\lor\neg q)\land(\neg p\lor q)), (p\lor q), (p\lor\neg q), (\neg p\lor q), (\neg p\lor\neg q), (\Upsilon).$$

Contents

Implication Algebra

- We apply the support tool to show that some axioms with three free variables in several implication algebras [Abbott, 1976, Georgacarakos, 1980, Hardegree, 1981, Chajda et al., 2001] are valid.
- Before that, we briefly explain what implication algebras are.
  - In Boolean lattices, the only implication is $p \to q = \neg p \vee q$.
  - However, in orthomodular lattices, there are distinct implications defined as follows:

$$p \rightsquigarrow q = \neg p \vee (p \wedge q), \quad p \rightarrowtail q = (\neg p \wedge \neg q) \vee q,$$
$$p \twoheadrightarrow q = ((p \wedge q) \vee (\neg p \wedge q)) \vee (\neg p \wedge \neg q).$$

  - **Implication algebra**: algebra which only operator is the implication (and $\curlywedge$).
  - Some implication algebras for orthomodular lattices have been proposed:
    1. Quasi-implication algebra [Hardegree, 1981]
    2. Ortho-implication algebra [Abbott, 1976]
    3. Orthomodular implication algebra [Chajda et al., 2001]
    4. Sasaki implication algebra [Georgacarakos, 1980]
    5. Dishkant implication algebra [Georgacarakos, 1980]
    6. Relevance implication algebra [Georgacarakos, 1980]

Axioms in Implication Algebra

- The axiom (Q2) of quasi-implication algebra [Hardegree, 1981]:

$$(p \rightsquigarrow q) \rightsquigarrow (p \rightsquigarrow r) = (q \rightsquigarrow p) \rightsquigarrow (q \rightsquigarrow r).$$

- The axiom (O2) of ortho-implication algebra [Abbott, 1976]:

$$p \rightarrowtail ((q \rightarrowtail p) \rightarrowtail r) = p \rightarrowtail r.$$

- The axiom (O5) of orthomodular implication algebra [Chajda et al., 2001]:

$$(((p \rightarrowtail q) \rightarrowtail q) \rightarrowtail r) \rightarrowtail (p \rightarrowtail r) = \curlyvee$$

- The axiom (O6) of orthomodular implication algebra [Chajda et al., 2001]:

$$((((((((p \rightarrowtail q) \rightarrowtail q) \rightarrowtail r) \rightarrowtail r) \rightarrowtail r) \rightarrowtail p) \rightarrowtail p) \rightarrowtail r) \rightarrowtail p) \rightarrowtail p$$
$$= (((p \rightarrowtail q) \rightarrowtail q) \rightarrowtail r) \rightarrowtail r$$

- The axiom (J4) of Sasaki implication algebra [Georgacarakos, 1980]:

$$p \rightsquigarrow ((p \rightsquigarrow ((q \rightsquigarrow ((q \rightsquigarrow r) \rightsquigarrow \curlywedge)) \rightsquigarrow \curlywedge)) \rightsquigarrow \curlywedge)$$
$$= r \rightsquigarrow ((r \rightsquigarrow ((p \rightsquigarrow ((p \rightsquigarrow q) \rightsquigarrow \curlywedge)) \rightsquigarrow \curlywedge)) \rightsquigarrow \curlywedge).$$

- The axiom (K5) of Dishkant implication algebra [Georgacarakos, 1980]:

$$(((p \rightarrowtail q) \rightarrowtail q) \rightarrowtail r) \rightarrowtail r = (p \rightarrowtail ((q \rightarrowtail r) \rightarrowtail r)) \rightarrowtail ((q \rightarrowtail r) \rightarrowtail r).$$

- The axiom (L6) of relevance implication algebra [Georgacarakos, 1980]:

$$(((p \twoheadrightarrow q) \twoheadrightarrow q) \twoheadrightarrow r) \twoheadrightarrow r = (p \twoheadrightarrow ((q \twoheadrightarrow r) \twoheadrightarrow r)) \twoheadrightarrow ((q \twoheadrightarrow r) \twoheadrightarrow r).$$

Experimental Result

- The axioms are verified by our tool automatically.
- Our tool is publicly available at `https://github.com/canhminhdo/FOM`.

Target Axiom	Time
The axiom (Q2) in [Hardegree, 1981]	1,213ms
The axiom (O2) in [Abbott, 1976]	736ms
The axiom (O5) in [Chajda et al., 2001]	705ms
The axiom (O6) in [Chajda et al., 2001]	716ms
The axiom (J4) in [Georgacarakos, 1980]	715ms
The axiom (K5) in [Georgacarakos, 1980]	723ms
The axiom (L6) in [Georgacarakos, 1980]	6d:19h:40m

Contents

Conclusion

- Using a reachability analysis, we have described how to develop the support tool for checking the word problem with three or more generators for free orthomodular lattices.
- The existing tools [Megill and Pavičić, 2001, Hyčko, 2005] cannot deal with terms that consist of three or more free variables.
- We have conducted some case studies with the support tool to verify various complex axioms that existing tools cannot verify.

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